Derivation of Optimum Winding Thickness for Duty Cycle Modulated Current Waveshapes

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Abstract - **Increased switching frequencies in magnetic components have resulted in renewed attention to the problem of proximity effect losses in layered transformer windings. The ideal situation is to design at the point of minimum a.c. winding resistance. This paper provides a unified approach which gives exact a.c. resistance formulas for pulsed, rectangular and triangular waveforms, with variable duty cycle. In every case an approximation to the a.c. resistance versus layer thickness curve is derived and the optimum point can be found with a simple calculation involving the number of harmonics (related to rise time), the duty cycle and the number of layers. This process yields a result that is at least as accurate as reading the point from a generated graph (without the considerable effort involved in generating the graph).**

NOMENCLATURE

a, b	Constants in approximation formulas
d	Thickness of foil or layer.
D	Duty cycle.
f	Frequency of current waveform in Hz.
I_{dc}	Average value of current.
I_n	R.M.S. value of the n^{th} harmonic.
I_o	Peak positive value of current.
I_{rms}	R.M.S. value of current waveform.
k_{p_n}	Proximity effect factor at n^{th} harmonic.
n	Harmonic number.
N	Maximum number of harmonics.
p	Number of layers.
r_o	Radius of bare wire in wire-wound winding.
R_{dc}	d.c. resistance of a winding.
R_{eff}	Effective a.c. resistance of a winding.
R_δ	d.c. resistance of a winding of thickness δ_o.
t_r	Rise time.
T	Period of current waveform.
δ_n	Skin depth at the n^{th} harmonic.
δ_o	Skin depth at fundamental frequency, f.
Δ	d/δ.

I. INTRODUCTION

The increased switching frequencies in magnetic components have resulted in renewed attention to the problem of proximity effect losses in layered transformer windings. The ideal situation is to design at the point of minimum a.c. winding resistance in order to minimize these losses. Dowell [1] gives an a.c. resistance factor for sinusoidal currents, and Carsten [2] deals with pulsed, triangular and square waveshapes with 50% or 100% duty cycle. Perry [3] deals with multilayer windings with variable layer thickness for sinusoidal waveforms. Venkatraman [4] refined the pulsed waveform approach by introducing a variable duty cycle. Vandelac and Ziogas [5] introduced an alternative graphical approach based on MMF diagrams. In all cases, the optimum point is found by plotting the a.c. winding resistance against layer thickness and the optimum layer thickness is read from the graph. This is not a straightforward task since the Fourier Series of the waveform is required, and the a.c. resistance at each frequency component must be calculated. The a.c. analysis in [2] to [5] is based on Dowell's formula [1] which is a one-dimensional plate approximation to the field solution for a cylindrical winding [6], this approach is justified when the thickness of the layer is less than 5% of the radius of curvature.

This paper provides a unified approach which gives exact formulas for bipolar rectangular, triangular and sinusoidal waveforms and their rectified equivalents, with variable duty cycle, as illustrated in Table 1. In every case an approximation to the a.c. resistance versus layer thickness curve is derived and the optimum point can be found with a simple calculation involving the number of harmonics (related to rise time), duty cycle and number of layers. This process yields a result that is at least as accurate as reading the point from a generated graph (without the considerable effort involved in generating the graph).

These new formulas have been derived for a wide range of waveshapes and are given in terms of the duty-cycle, number of layers and harmonics, the time previously required to plot an endless supply of resistance-thickness graphs, for cases

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where these variables are changing, is eliminated. This paper gives an example of a push-pull converter and shows that a multilayer foil winding is superior to a round conductor configuration.

II. Waveform Analysis

Formulas for the optimum winding or layer thickness are unique to each current waveform illustrated in Table 1, but the same method is followed in each case. A pulsed (or rectified rectangular) waveform with variable duty cycle as encountered in a push-pull converter is analyzed to illustrate the methodology.

The current waveform shown in Fig. 1 is representative of that in the winding of a forward or push-pull converter. The physical layout of a typical winding is illustrated in Fig. 2, round conductors are converted to equivalent layers as shown. The waveform in Fig. 1 is an even function about a zero point at the center of the pulse. The Fourier Series is

$$i(t) = a_0 + \sum_{n=1}^{\infty} a_n \mathrm{Cos}\left(\frac{2n\pi}{T}t\right) \qquad (1)$$

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} i(t)dt = I_o D$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} i(t)\mathrm{Cos}\left(\frac{2n\pi}{T}t\right)dt = \frac{2I_o}{n\pi}\mathrm{Sin}(n\pi D)$$

yielding

$$i(t) = I_o D + \sum_{n=1}^{\infty}\frac{2I_o}{n\pi}\mathrm{Sin}(n\pi D)\mathrm{Cos}\left(\frac{2n\pi}{T}t\right) \qquad (2)$$

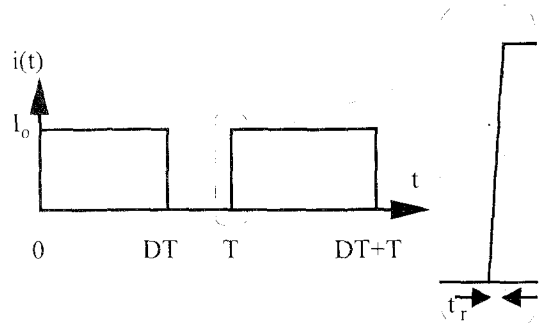

Fig. 1: Pulsed current waveform with a duty-cycle of D and a rise time t_r.

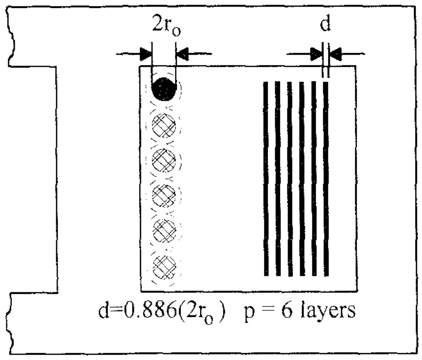

Fig. 2: Equivalent layers in a wire wound winding.

The average value of current is $I_{dc} = I_o D$. The R.M.S. value of current is $I_o\sqrt{D}$ The R.M.S. value of the n^{th} harmonic is

$$I_n = \frac{1}{\sqrt{2}}\left[\frac{2I_o}{n\pi}\mathrm{Sin}(n\pi D)\right] = \frac{\sqrt{2}I_o}{n\pi}\mathrm{Sin}(n\pi D) \qquad (3)$$

III. Proximity Effect Factors

The pulsed waveform in Fig. 1 is not an ideal case as there is a rise time and fall time associated with it so that a finite number of harmonics are required. Typically, the upper limit on the number of harmonics is

$$N = \frac{35}{t_r\%} \qquad (4)$$

where t_r is the percentage rise time as shown in Fig. 1 and N is odd. For example, a 2.5% rise time would give N = 13.

The total power loss is $P = R_{eff}I_{rms}^2$ which is made up of the d.c. component and N harmonics:

$$P = R_{dc}I_{dc}^2 + \sum_{n=1}^{N} R_{ac_n}I_n^2 = R_{dc}I_{dc}^2 + R_{dc}\sum_{n=1}^{N} k_{p_n}I_n^2 \qquad (5)$$

where R_{ac_n} is the a.c. resistance at the n^{th} harmonic and R_{dc} is the d.c. resistance of a foil winding of thickness d. k_{p_n} is the proximity effect factor due to the n^{th} harmonic [1]:

$$k_{p_n} = \Delta_n\left[\frac{\mathrm{Sinh}(2\Delta_n)+\mathrm{Sin}(2\Delta_n)}{\mathrm{Cosh}(2\Delta_n)-\mathrm{Cos}(2\Delta_n)} + \frac{2(p^2-1)}{3}\,\frac{\mathrm{Sinh}(\Delta_n)-\mathrm{Sin}(\Delta_n)}{\mathrm{Cosh}(\Delta_n)+\mathrm{Cos}(\Delta_n)}\right] \qquad (6)$$

where p is the number of layers required, and Δ_n is equal to the thickness of a layer, d, divided by δ_n, the skin depth at the n^{th} harmonic. Defining δ_o as the skin depth at the fundamental frequency of the pulsed waveform, δ_n and Δ_n are given by

$$\delta_n = \frac{\delta_o}{\sqrt{n}} \quad \Rightarrow \quad \Delta_n = \frac{d}{\delta_n} = \sqrt{n}\Delta \tag{7}$$

where

$$\Delta = \frac{d}{\delta_o}$$

Since $P = R_{eff} I_{rms}^2$, equation (5) can be rearranged to yield

$$\frac{R_{eff}}{R_{dc}} = \frac{I_{dc}^2 + \sum_{n=1}^{\infty} k_{p_n} I_n^2}{I_{rms}^2}$$

$$= \frac{I_o^2 D^2 + \frac{2 I_o^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{Sin}^2(n\pi D) k_{p_n}}{I_o^2 D} \tag{8}$$

$$= D + \frac{2}{\pi^2 D} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{Sin}^2(n\pi D) k_{p_n}$$

Define R_δ as the d.c. resistance of a foil of thickness δ_o such that

$$\frac{R_\delta}{R_{dc}} = \frac{d}{\delta_o} = \Delta \quad \Rightarrow \quad \frac{R_{eff}}{R_{dc}} = \Delta \frac{R_{eff}}{R_\delta} \tag{9}$$

Evidently, a plot of R_{eff} / R_δ versus Δ would have the same shape as a plot of R_{eff} versus d at a given frequency. A 3-D plot of R_{eff} / R_δ versus Δ with p, the number of layers, on the third axis is shown in Fig. 3.

For a given number of layers there is an optimum point, Δ_{opt}, at which the a.c. resistance is minimum. These optimum points lie on the line marked minima in the graph, and the corresponding optimum thickness is given by

$$d_{opt} = \Delta_{opt} . \delta_o \tag{10}$$

Substituting for k_{p_n} given by (6) in (8) yields an expression for R_{eff} / R_{dc}:

$$\frac{R_{eff}}{R_{dc}} = D + \frac{2\Delta}{\pi^2 D} \sum_{n=1}^{N} \frac{\mathrm{Sin}^2(n\pi D)}{n^{\frac{3}{2}}} \cdot \left[\frac{\mathrm{Sinh}(2\sqrt{n}\Delta) + \mathrm{Sin}(2\sqrt{n}\Delta)}{\mathrm{Cosh}(2\sqrt{n}\Delta) - \mathrm{Cos}(2\sqrt{n}\Delta)} + \frac{2(p^2-1)}{3} \frac{\mathrm{Sinh}(\sqrt{n}\Delta) - \mathrm{Sin}(\sqrt{n}\Delta)}{\mathrm{Cosh}(\sqrt{n}\Delta) + \mathrm{Cos}(\sqrt{n}\Delta)} \right] \tag{11}$$

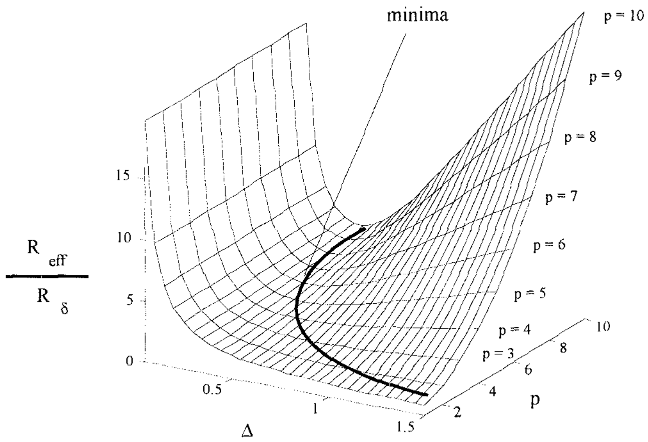

Fig. 3: Plot of R_{eff} / R_δ versus Δ, for N = 13 harmonics, D = 50% duty-cycle.

IV. Approximate Analysis

A. Taylor Series

The following general approximations to y_1 and y_2 can be made by expanding the trigonometric functions using Taylor's series and limiting them to three terms:

$$y_1 = \frac{\mathrm{Sinh}(2\Delta) + \mathrm{Sin}(2\Delta)}{\mathrm{Cosh}(2\Delta) - \mathrm{Cos}(2\Delta)} \approx \frac{1}{\Delta} + \frac{\Delta^3}{a}$$
$$y_2 = \frac{\mathrm{Sinh}(\Delta) - \mathrm{Sin}(\Delta)}{\mathrm{Cosh}(\Delta) + \mathrm{Cos}(\Delta)} \approx \frac{\Delta^3}{b} \tag{12}$$

The unknown parameters a and b are found to be 7.5 and 6 respectively from the Taylor's series analysis.

B. Regression Analysis

The values of a and b may be further refined using a nonlinear curve fitting method, which fits a user-defined model to data points. A model is linear in its parameters if the parameters are all added or multipled times a variable. However, this is not the case in the above approximations, and the nonlinear estimation method developed by Marquadt and Levenberg [7], as detailed in the appendix, is used. Applying this method to the two approximations yields normal equations for a and b as follows:

$$a = \frac{\sum_{u=1}^{N_d} \Delta_u^6}{\sum_{u=1}^{n} \left(y_{1_u} \Delta_u^3 - \Delta_u^2 \right)}, \quad b = \frac{\sum_{u=1}^{N_d} \Delta_u^6}{\sum_{u=1}^{n} y_{2_u} \Delta_u^3} \tag{13}$$

where Δ is the independent variable, N_d is the number of data points taken, and y_1 and y_2 are the corresponding dependent variables in (12). For Δ between 0.1 and 1.0, a is 11.571 and b is 6.182.

The proximity effect factor k_{p_n} in (12) may be approximated as:

$$k_{p_n} \approx 1+\frac{\Delta_n^{\,4}}{a}+\frac{2(p^2-1)}{3}\frac{\Delta_n^{\,4}}{b} = 1+\frac{1}{3}\left[\frac{2p^2-2}{b}+\frac{3}{a}\right]\Delta_n^{\,4} \tag{14}$$

Substituting this expression into (8) using (7) and (9) yields

$$\frac{R_{eff}}{R_\delta}=\frac{D}{\Delta}+\frac{2}{\pi^2 D}\sum_{n=1}^{N}\frac{\mathrm{Sin}^2(n\pi D)}{n^2\Delta}\times\left(1+\left[\frac{2p^2-2}{3b}+\frac{1}{a}\right]\left(\sqrt{n}\Delta\right)^4\right) = \frac{D+\dfrac{2}{\pi^2 D}\displaystyle\sum_{n=1}^{N}\frac{\mathrm{Sin}^2(n\pi D)}{n^2}}{\Delta}+\frac{2}{\pi^2 D}\sum_{n=1}^{N}\mathrm{Sin}^2(n\pi D)\left[\frac{2p^2-2}{3b}+\frac{1}{a}\right]\Delta^3 \tag{15}$$

The derivative of (15) with respect to Δ is used to calculate the optimum value of Δ:

$$\frac{d\left(\dfrac{R_{eff}}{R_\delta}\right)}{d\Delta}=-\frac{\left[D+\dfrac{2}{\pi^2 D}\displaystyle\sum_{n=1}^{N}\frac{\mathrm{Sin}^2(n\pi D)}{n^2}\right]}{\Delta^2}+\left(\frac{2}{\pi^2 D}\sum_{n=1}^{N}\mathrm{Sin}^2(n\pi D)\left[\frac{2p^2-2}{b}+\frac{3}{a}\right]\right)\Delta^2 \tag{16}$$

Setting this equal to zero gives

$$\Delta_{opt}=\sqrt[4]{\frac{D+\dfrac{2}{\pi^2 D}\displaystyle\sum_{n=1}^{N}\frac{\mathrm{Sin}^2(n\pi D)}{n^2}}{\left[\dfrac{2p^2-2}{b}+\dfrac{3}{a}\right]\dfrac{2}{\pi^2 D}\displaystyle\sum_{n=1}^{N}\mathrm{Sin}^2(n\pi D)}} \tag{16}$$

If D = 0.5, then the formula for Δ_{opt} is given by

$$\Delta_{opt}=\sqrt[4]{\frac{0.5+\dfrac{4}{\pi^2}\displaystyle\sum_{n=1,\,odd}^{N}\frac{1}{n^2}}{\left[\dfrac{2p^2-2}{b}+\dfrac{3}{a}\right]\dfrac{4}{\pi^2}\left(\dfrac{N+1}{2}\right)}} \tag{17}$$

For large N (short rise time) and with a = 11.571, b = 6.182, Δ_{opt} is given by

$$\Delta_{opt}=\frac{1}{\sqrt[4]{(0.1312p^2-0.0261)\left(\dfrac{N+1}{2}\right)}} \tag{18}$$

Substituting (16) in (15) and then in (9) yields an approximation for R_{eff}/R_{dc}:

$$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt}=\frac{4D}{3}+\frac{8}{3\pi^2 D}\sum_{n=1}^{N}\frac{1}{n^2}\mathrm{Sin}^2(n\pi D) \tag{19}$$

In general, R_{eff}/R_{dc} is in the range 1.3 to 1.4 at the optimum point.

Formulas for other waveshapes are given in Table 1.

V. DESIGN EXAMPLE: PUSH-PULL CONVERTER

A push-pull converter is shown in Fig. 4 and its associated waveforms are illustrated in Fig. 5. The current waveform in each primary winding may be approximated to the pulsed waveform of Fig. 1, with the ripple neglected.

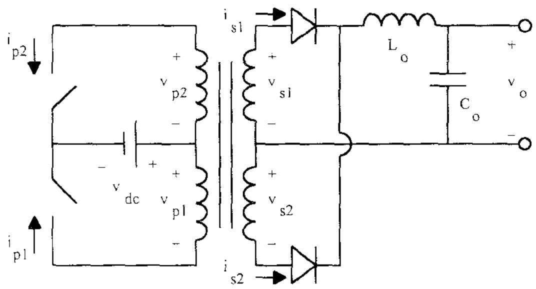

Fig. 4: Push-pull converter, circuit.

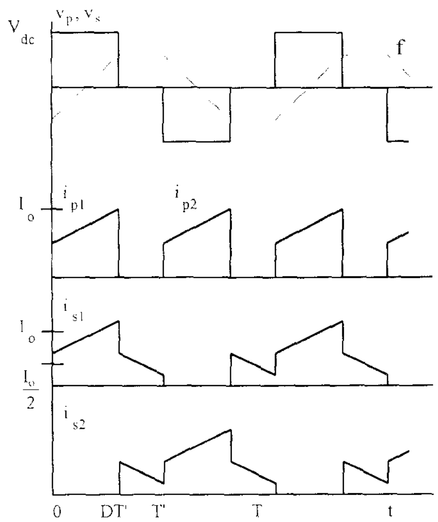

Fig. 5: Push-pull converter, waveforms.

Take D = 0.5, p (number of foil layers) = 6, t_r (rise time) = 2.5%, and f = 50 kHz. For t_r = 2.5%, take 13 harmonics.

$$\Delta_{opt} = \frac{1}{\sqrt[4]{([13+1]/2)(0.1312 \times 6^2 - 0.0261)}} = 0.42$$

$$\delta_o = \frac{66}{\sqrt{f}} = \frac{66}{\sqrt{50 \times 10^3}} = 0.295 \text{ mm}$$

The optimum foil thickness is $d_{opt} = \Delta_{opt}\delta_o = 0.42 \times 0.295 = 0.12$ mm. The a.c. resistance is found from (19):

$$\frac{R_{eff}}{R_{dc}} = 1.314$$

Alternatively, the winding could be constructed with a single layer of round conductors; assuming a window height of 30 mm, a 2.14 mm diameter of bare copper wire has the same copper area as a 0.12 × 30 mm foil, in this case p = 1, d = 0.886(2.14) = 1.896, Δ = 1.896/0.295 = 6.427, and

$$\frac{R_{eff}}{R_{dc}} = 4.203$$

Evidently in this case, the choice of a foil is vastly superior.

VI. Conclusions

The paper describes a general procedure to calculate a.c. resistance of multilayer windings for general waveshapes encountered in switching mode power supplies. Variable duty cycle is an integral part of the procedure. In each case, a simple and accurate approximation is established so that the optimum layer thickness may be found from knowledge of the number of layers, number of harmonics (related to rise time) and duty cycle.

Appendix

The Marquadt and Levenberg method represents a compromise between linearisation (or Taylor's series) methods and the steepest descent method and appears to combine the best features of both while avoiding their most serious limitations.

Nonlinear models all have the general form $y = f(x, a, b, \ldots) + \varepsilon$ where y is the dependent variable, x is one or more independent variables, a, b, … are the unknown parameters to be estimated, f() is the nonlinear function of the unknown parameters and independent variables, and ε is the error term.

Marquadt's method can be used to estimate the parameters a, b, … of the nonlinear model using given data points. The residual sum of squares formula for the model given above can be written as

$$\sum e^2 = \sum_{u=1}^{n} (y_u - f(x_u, a, b))^2$$

where (x_u, y_u) are the corresponding data point pairs (independent variable, dependent variable) for u from 1 to n, the total number of data points, and $f(x_u, a, b, \ldots)$ is the nonlinear function evaluated at its corresponding x_u value.

The unknowns a, b, … are to be chosen to make $\sum e^2$ a minimum, so that the derivatives of $\sum e^2$ with respect to a, b, … must vanish. Therefore,

$$\frac{\partial\left(\sum e^2\right)}{\partial a} = 2\sum\left(\frac{\partial e}{\partial a} e\right) = 0$$

$$\frac{\partial\left(\sum e^2\right)}{\partial b} = 2\sum\left(\frac{\partial e}{\partial b} e\right) = 0$$

$$\vdots$$

Normal equations for the unknown parameters are then derived from these equations.

Acknowledgment

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References

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[3] M.P. Perry, "Multiple Layer Series Connected Winding Design for Minimum Losses", *IEEE Trans. on Power Apparatus and Systems*, vol. 98, no. 1, pp. 116-123, January 1979.

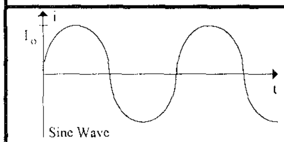
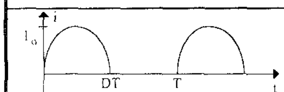
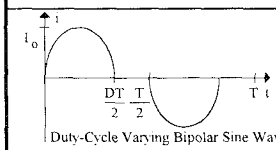
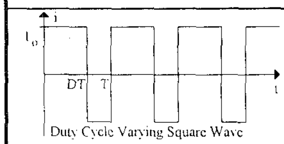
[4] P.S. Venkatraman, "Winding Eddy Current Losses in Switch Mode Power Transformers Due to Rectangular Wave Currents", *Proc. of Powercon 11*, section A-1, pp. 1-11, 1984.

[5] J. Vandelac, P.D. Ziogas, "A Novel Approach for Minimizing High-Frequency Transformer Copper Losses", *IEEE Trans. on Power Electronics*, vol. 3, no. 3, pp. 266-276, July 1988.

[6] E. Bennett, S.C. Larson, "Effective Resistance to Alternating Currents of Multilayer Windings", *Trans. of AIEE*, vol. 59, pp. 1010-1016, 1940.

[7] Draper, N.R., Smith, H., *Applied Regression Analysis*, New York: John Wiley and Sons, 1966.

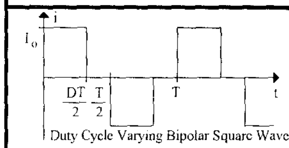
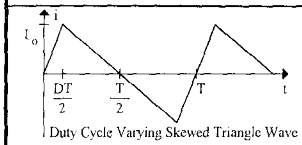
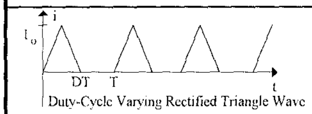
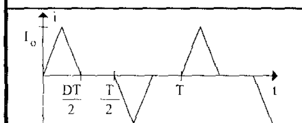
Table 1: Approximation formulas for the optimum thickness and ac to dc resistance ratio of a winding for various waveforms, $a = 7.5$, $b = 6$, $\Psi = \frac{2p^2 - 2}{b} + \frac{3}{a}$.

Current Waveform and Corresponding Fourier Series	**Formulas for $[R_{eff}/R_{dc}]_{opt}$**	**Formulas for Δ_{opt}**
 Sine Wave	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{4}{3}$	$\Delta_{opt} = \sqrt[4]{\frac{1}{\Psi}}$
 Duty-Cycle Varying Rectified Sine Wave D = 1 for Full Wave Rectification D = 0.5 for Half Wave Rectification $i(t) = \frac{2DI_o}{\pi} + \sum_{n=1}^{\infty} \frac{4DI_o}{\pi} \left\{ \frac{Cos(n\pi D)}{\left(1 - 4n^2D^2\right)} \right\} Cos(n\omega t)$ ♣	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{32D}{3\pi^2} + \frac{64D}{3\pi^2} \sum_{n=1}^{N} \left\{ \frac{Cos^2(n\pi D)}{(1 - 4n^2D^2)^2} \right\}$ ♣	$\Delta_{opt} = \sqrt[4]{\frac{\frac{1}{2} + \sum_{n=1}^{N} \left\{ \frac{Cos^2(n\pi D)}{\left(1 - 4n^2D^2\right)^2} \right\}}{\Psi \sum_{n=1}^{N} \left\{ \frac{Cos^2(n\pi D)}{\left(1 - 4n^2D^2\right)^2} \right\} n^2}}$ ♣
 Duty-Cycle Varying Bipolar Sine Wave $i(t) = \sum_{n=1,\,odd}^{\infty} \frac{4DI_o}{\pi} \left\{ \frac{Cos\left(\frac{n\pi D}{2}\right)}{(1 - n^2D^2)} \right\} Cos(n\omega t)$ ♦	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{64D}{3\pi^2} \sum_{n=1,\,odd}^{N} \left\{ \frac{Cos^2\left(\frac{n\pi D}{2}\right)}{(1 - n^2D^2)^2} \right\}$ ♦	$\Delta_{opt} = \sqrt[4]{\frac{\sum_{n=1,\,odd}^{N} \left\{ \frac{Cos^2\left(\frac{n\pi D}{2}\right)}{\left(1 - n^2D^2\right)^2} \right\}}{\Psi \sum_{n=1,\,odd}^{N} \left\{ \frac{Cos^2\left(\frac{n\pi D}{2}\right)}{\left(1 - n^2D^2\right)^2} \right\} n^2}}$ ♦
 Duty Cycle Varying Square Wave $i(t) = I_o(2D - 1) + \sum_{n=1}^{\infty} \frac{4I_o}{n\pi} Sin(n\pi D) Cos(n\omega t)$	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{4(2D - 1)^2}{3} + \frac{32}{3\pi^2} \sum_{1}^{N} \frac{Sin^2(n\pi D)}{n^2}$	$\Delta_{opt} = \sqrt[4]{\frac{(2D - 1)^2 + \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{Sin^2(n\pi D)}{n^2}}{\Psi - \frac{8}{\pi^2} \sum_{n=1}^{N} Sin^2(n\pi D)}}$
 Duty Cycle Varying Rectified Square Wave $i(t) = I_oD + \sum_{n=1}^{\infty} \frac{2I_o}{n\pi} Sin(n\pi D) Cos(n\omega t)$	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{4D}{3} + \frac{8}{3\pi^2 D} \sum_{n=1}^{N} \frac{1}{n^2} Sin^2(n\pi D)$	$\Delta_{opt} = \sqrt[4]{\frac{D + \frac{2}{\pi^2 D} \sum_{n=1}^{N} \frac{Sin^2(n\pi D)}{n^2}}{\Psi - \frac{2}{\pi^2 D} \sum_{n=1}^{N} Sin^2(n\pi D)}}$

♣ For $n = k = 1/2D \in \mathbf{N}$ (the set of natural numbers), the expressions in {curly brackets} are replaced by $\frac{\pi^2}{16}$.

♦ For $n = k = 1/D \in \mathbf{N}$ (the set of natural numbers), the expressions in {curly brackets} are replaced by $\frac{\pi^2}{16}$.

Table 1 (continued)

Current Waveform and Corresponding Fourier Series	Formulas for $[R_{eff}/R_{dc}]_{opt}$	Formulas for Δ_{opt}
 Duty Cycle Varying Bipolar Square Wave $i(t) = \sum_{n=1,\,odd}^{\infty} \frac{4I_o}{n\pi} Sin\left(\frac{n\pi D}{2}\right) Cos(n\omega t)$	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{32}{3\pi^2 D} \sum_{n=1,\,odd}^{N} \frac{1}{n^2} Sin^2\left(\frac{n\pi D}{2}\right)$	$\Delta_{opt} = \sqrt[4]{\frac{\sum_{n=1,\,odd}^{N} Sin^2\left(\frac{n\pi D}{2}\right)\frac{1}{n^2}}{\Psi \sum_{n=1,\,odd}^{N} Sin^2\left(\frac{n\pi D}{2}\right)}}$
 Duty Cycle Varying Skewed Triangle Wave $i(t) = \sum_{n=1}^{\infty} \frac{2I_o Sin(n\pi D))}{\pi^2 n^2 D(1-D)}$	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{8}{\pi^4 (D-D^2)^2} \sum_{n=1}^{N} \frac{Sin^2(n\pi D)}{n^4}$	$\Delta_{opt} = \sqrt[4]{\frac{\sum_{n=1}^{N} \frac{Sin^2(n\pi D)}{n^4}}{\Psi \sum_{n=1}^{N} \frac{Sin^2(n\pi D)}{n^2}}}$
 Duty-Cycle Varying Rectified Triangle Wave $i(t) = \frac{I_o D}{2} + \sum_{n=1}^{\infty} \frac{4I_o}{\pi^2 n^2 D} Sin^2\left(\frac{n\pi D}{2}\right) Cos(n\omega t)$	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = D + \frac{32}{\pi^4 D^3} \sum_{n=1}^{N} \frac{1}{n^4} Sin^4\left(\frac{n\pi D}{2}\right)$	$\Delta_{opt} = \sqrt[4]{\frac{1 + \frac{32}{\pi^4 D^4} \sum_{n=1}^{N} \frac{1}{n^4} Sin^4\left(\frac{n\pi D}{2}\right)}{\Psi \frac{32}{\pi^4 D^4} \sum_{n=1}^{N} \frac{1}{n^2} Sin^4\left(\frac{n\pi D}{2}\right)}}$
 Duty Cycle Varying Bipolar Triangle Wave $i(t) = \sum_{n=1,\,odd}^{\infty} \frac{8I_o}{\pi^2 n^2 D}\left(1 - Cos\left(\frac{nD\pi}{2}\right)\right) Cos(n\omega t)$	$\left[\frac{R_{eff}}{R_{dc}}\right]_{opt} = \frac{128}{\pi^4 D^3} \sum_{n=1,\,odd}^{N} \frac{1}{n^4}\left(1 - Cos\left(\frac{n\pi D}{2}\right)\right)^2$	$\Delta_{opt} = \sqrt[4]{\frac{\sum_{n=1,odd}^{N}\left(1 - Cos\left(\frac{n\pi D}{2}\right)\right)^2 \frac{1}{n^4}}{\Psi \sum_{n=1,\,odd}^{N}\left(1 - Cos\left(\frac{n\pi D}{2}\right)\right)^2 \frac{1}{n^2}}}$